SENIOR PROJECT

Elastic Buckling and Free Vibration Analysis of Nonlocal Euler-Bernoulli Beams

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Abstract

This senior project presents the elastic buckling and free vibration analysis of beams by taking the influence of nonlocal elasticity into account. In the modeling, linearized Euler-Bernoulli beam theory together with Eringen nonlocal theory is utilized to describe the static and dynamic responses of the beam and formulate the key governing equations. An integral form of the nonlocal constitutive model is adopted to in the present study to better handle a body of finite dimensions. Galerkin approximation is implemented to construct the numerical solutions and standard Gaussian quadrature is employed to evaluate all involved integrals. A power method together with Rayleigh quotient is adopted to efficiently determine the buckling load, fundamental frequency and the corresponding mode shape. After sufficiently tested with available benchmark solutions, the proposed technique is then applied to thoroughly investigate the influence of nonlocal effect and the size dependency on the buckling and free vibration characteristics.

Keywords: Nonlocal elasticity, Euler-Bernoulli Beam, Eringen integral model, Galerkin approximation

1. Introduction

Nowadays, nanotechnology has been widely use in various applications in our daily life due to their large Young's modulus, yield strength, flexibility, and conductivity properties (Zhang et al. [1]) like nanostructures (e.g. nanotubes, nanorods, nanowires, etc.) in NEMS (e.g. sensors for force detection, mass spectrometry, inertial imaging, calorimetry, and charge sensing) and MEMS (e.g. MEMS pressure sensors, MEMS inertial sensors). Learning and understanding its properties are necessary and noteworthy before applying it. Thus, this induces us to have more interest in investigating nanostructure's behaviors which are buckling and vibration characteristics.

By investigating a number of papers, it has been shown that many researches pay attention to explore both static and dynamic behavior of nanobeams by using continuum modelling techniques due to its simplicity and effective computation, whereas performing an experiments are expensive and atomistic simulation methods require huge amount of computation effort for nanostructures.

Instead of original integral form, the theory is generally used in its differential form due to its simplicity. For example, Reddy [2], Aydogdu [3], Thai [4], and Eltaher et al. [5]. Nevertheless, many studies pointed out the paradox appeared in the case of cantilever beams. (Challamel and Wang [6], Murmu and Pradhan [7], and Lee and Chang [8]).Therefore, recently, Fernández-Sáez et al. [9] have proposed a general method to solve the original integral model, and compared their results with the differential form for static bending case. Consequently, they indicated that using differential form is not convenient. Accordingly, Tuna and Kirca [10] solved the original integral model for static bending of Euler-Bernoulli beam and Timoshenko beams and found that differential form of Eringen nonlocal model (Eringen [11]) is unable to address the effects of small length scales except for simply supported case, while integral form predicts effect of nonlocal parameter acculately.

Thus, later studies tend to apply integral form instead of differential form in analysis of buckling and vibration of nanobeams. Tuna and Kirca [12] proposed an exact solution of Eringen's nonlocal integral model for bending of Euler-Bernoulli and Timoshenko beam and then presented an exact solution of Eringen's nonlocal integral model for buckling and free vibration of Euler-Bernoulli beam in 2016. After that they utilized the integral form of Eringen nonlocal model in analysis of static bending, linear buckling and free vibration of nanobeam structures by using finite element method in 2017 .

According to many pioneering researches,there are various continuum theories that include small-size effect such as the strain gradient theory (Nix and Gao [13]), couple stress theory (Hadjesfandiari and Dargush [14]), modified couple stress theory (Asghari et al. [15]) and nonlocal elasticity theory (Eringen [11], Eringen and Edelen [16], Eringen [17]). Among various continuum theories, Eringen's nonlocal of elasticity theory has been popularly used to address the small size effects. Unlike the local theories which assume that the stress at a point is a function of strain at that point, the theory states that stress at a point is influenced by the curvature of all points in the domain through a kernel function.

This present investigation intends to examine both static and dynamic behaviors of perfectly straight, prismatic beam with fixed-free end conditions. Only the response in the elastic regime is of interest and the influence of axial and shear deformation and the rotational inertia is fully negligible. The considerable behaviors includes buckling load, buckling shape, fundamental natural frequency and amplitude of deflection by the Galerkin's approximation solution technique which applied the nonlocal elasticity constitutive equations in solving.

2. Problem Formulation

2.1 Problem description

Consider a one-dimentional, isotropic cantilever beams in nanoscale of length L subjected to a compression force P at the free end. The nanocantilever beams is made of a homogenous material of Young's modulus E and the cross section is A constant throughout the element. The problem statement, here, is to determine the buckling load, fundamental natural frequency as well as shape of buckling and vibration of nanocantilever beams.

2.2 Governing equations for buckling analysis

Eringen nonlocal theory together with adjusted area technique are adopted in representing relationship between static and maximum dynamic moment at point x which influenced by curvature at point ξ of entire domain through a kernel function as see in Eq.(4)-(5)

In order to eliminate complexity in computation, all parameters such that σ, A^0, M, P, v, M_0, ω and Λ are normalized denoted by length scale factor, kernel function, bending moment, axial force, deflection, maximum moment at point x related to time, natural frequency and amplitude of vibration respectively.

$$\bar{\xi} = \xi / l, \bar{x} = x / l, \bar{\sigma} = \sigma / l, \bar{A}^0 = A^0 l, \bar{M} = Ml / EI, \bar{P} = Pl^2 / EI, \bar{v} = v / l, \bar{M}_0 = M_0 l / EI, \bar{\omega} = \omega\sqrt{\frac{ml^4}{EI}}, \bar{\Lambda} = \Lambda / l \qquad (1)$$

By utilized basic equations, strong form equation can be established such that see in Eq.(2)-(3)

$$\frac{d^2\bar{M}}{d\bar{x}^2} + \bar{P}\frac{d^2\bar{v}}{d\bar{x}^2} = 0 \qquad (2)$$

$$\frac{d^2\overline{M_0}}{d\bar{x}^2}-\bar{\omega}^2\bar{\Lambda}\left(\bar{x}\right)=0 \tag{3}$$

$$\overline{M}\left(\bar{x}\right)=\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)\bar{\kappa}\left(\bar{\xi}\right)d\bar{\xi}+\left\{1-f\left(\bar{x}\right)\right\}\bar{\kappa}\left(\bar{x}\right) \tag{4}$$

$$\overline{M}_0\left(\bar{x}\right)=\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)\frac{d^2\bar{\Lambda}}{d\bar{\xi}^2}d\bar{\xi}+\left\{1-f\left(\bar{x}\right)\right\}\frac{d^2\bar{\Lambda}}{d\bar{x}^2} \tag{5}$$

$$f\left(\bar{x}\right)=\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\rho}\right|,\bar{\sigma}\right)d\bar{\rho} \tag{6}$$

A standard weight residual technique together with technique of integration by part are utilized to establish a weak-form equations as revealed in Eq.(7)-(8).

$$\int_0^1\frac{d^2\bar{w}}{d\bar{x}^2}\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)\frac{d^2\bar{v}}{d\bar{\xi}^2}d\bar{\xi}d\bar{x}+\int_0^1\left\{1-f\left(\bar{x}\right)\right\}\frac{d^2\bar{w}}{d\bar{x}^2}\frac{d^2\bar{v}}{d\bar{x}^2}d\bar{x}-\bar{P}\int_0^1\frac{d\bar{w}}{d\bar{x}}\frac{d\bar{v}}{d\bar{x}}d\bar{x}=0 \tag{7}$$

$$\int_0^1\frac{d^2\bar{w}}{d\bar{x}^2}\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)\frac{d^2\bar{\Lambda}}{d\bar{\xi}^2}d\bar{\xi}d\bar{x}+\int_0^1\left\{1-f\left(\bar{x}\right)\right\}\frac{d^2\bar{w}}{d\bar{x}^2}\frac{d^2\bar{\Lambda}}{d\bar{x}^2}d\bar{x}-\bar{\omega}^2\int_0^1\bar{w}\left(\bar{x}\right)\bar{\Lambda}\left(\bar{x}\right)d\bar{x}=0 \tag{8}$$

3. Numerical Implementation

In this section, Galerkin approximation is adopted to determine $\mathbf{K}$ by substituting both the weight function $\bar{w}\left(\bar{x}\right)$ and trial function $\bar{v}\left(\bar{x}\right)$ in weak form equation and transform the equation in form of eigen value equation as shown in Eq.(10)-(11)

$$\bar{w}\left(\bar{x}\right)=\sum_{i=1}^{N}\beta_i\phi_i\left(\bar{x}\right),\ \bar{v}\left(\bar{x}\right)=\sum_{j=1}^{N}\alpha_j\phi_j\left(\bar{x}\right),\ \bar{\Lambda}\left(\bar{x}\right)=\sum_{j=1}^{N}\alpha_j\phi_j\left(\bar{x}\right) \tag{9}$$

$$\left(\mathbf{K}-\bar{P}\mathbf{M_P}\right)\boldsymbol{\alpha}=\mathbf{0} \tag{10}$$

$$\left(\mathbf{K}-\bar{\omega}^2\mathbf{M_\omega}\right)\boldsymbol{\alpha}=\mathbf{0} \tag{11}$$

where $\boldsymbol{\alpha}$ denoted a vector containing degrees of freedom of the approximation and **K, M** are matrices defined by

$$\left[\mathbf{K}\right]_{ij}=K_{ij}=\int_0^1\frac{d^2\phi_i}{d\bar{x}^2}\int_0^1\bar{A}^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)\frac{d^2\phi_j}{d\bar{\xi}^2}d\bar{\xi}d\bar{x}+\int_0^1\left\{1-f\left(\bar{x}\right)\right\}\frac{d^2\phi_i}{d\bar{x}^2}\frac{d^2\phi_j}{d\bar{x}^2}d\bar{x} \tag{12}$$

$$\left[\mathbf{M_P}\right]_{ij}=M_{Pij}=\int_0^1\frac{d\phi_i}{d\bar{x}}\frac{d\phi_j}{d\bar{x}}d\bar{x} \tag{13}$$

$$\left[\mathbf{M_\omega}\right]_{ij}=M_{\omega ij}=\int_0^1\frac{d\phi_i}{d\bar{x}}\frac{d\phi_j}{d\bar{x}}d\bar{x} \tag{14}$$

Polynomial function is selected as a basis function $\phi_i\left(\bar{x}\right)$ which represents in Eq.(15). Gauss quadrature method is adopted to address the complexity of integral equation in order to determine $\mathbf{K}$ value in analysis of buckling and vibration characteristics.

$$\phi_i\left(\bar{x}\right)=\bar{x}^{i+1} \tag{15}$$

Buckling load, deflection, fundamental natural frequency and amplitude of vibration have been obtained by solving Eq.(10) and Eq.(11) using Matlab program developed by the authors.

4. Numerical Results

In this part, the small-scale effects on the buckling load and buckling shape of nanocantilever beams are obtained. In addition, effects of kernel's shape and size of influence zone on buckling load and fundamental natural frequency in case of nanocantilever beams are investigated.

Two properties parameters which affect both static and dynamic characteristics including kernel's shape as well as size of influence zone that depends on material properties are taken into account.

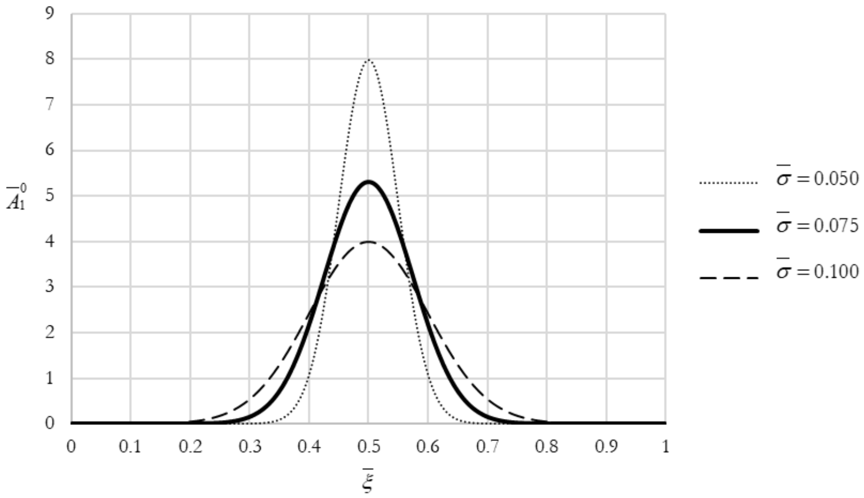

Figure.3 Kernel function versus location along the length of nanocantilever beams ($\bar{\xi}$) at the middle span ($\bar{x}$=0.5) by given $\bar{\sigma}$ equal to 0.050, 0.075 and 0.100

Two widely used of kernel's shape in investigating effects of two main parameters which are size of influence zone and shape are shown in Eq.(16)-(17) respectively.

$$\bar{A}_1^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\sigma}\right)=\frac{e^{-\left(\frac{\bar{x}-\bar{\xi}}{\sqrt{2}\bar{\sigma}}\right)^2}}{\sqrt{2\pi}\bar{\sigma}} \tag{16}$$

$$\bar{A}_2^0\left(\left|\bar{x}-\bar{\xi}\right|,\bar{\kappa}\right)=\frac{e^{-\left(\frac{\left|\bar{x}-\bar{\xi}\right|}{\bar{\kappa}}\right)}}{2\bar{\kappa}} \tag{17}$$

Figure.4 Illustrating kernel's shape of two nonlocal parameters $\bar{\sigma},\bar{\kappa}$ which set to be equal to 0.05, these parameters have different in size of influence zone. Hence, in order to adjusted the size, in following analysis, $\bar{\kappa}$ parameter is transformed in term of $\bar{\sigma}$ by relative equation which shown in Eq.(18)

$$\bar{\kappa}=\frac{\bar{\sigma}}{2} \tag{18}$$

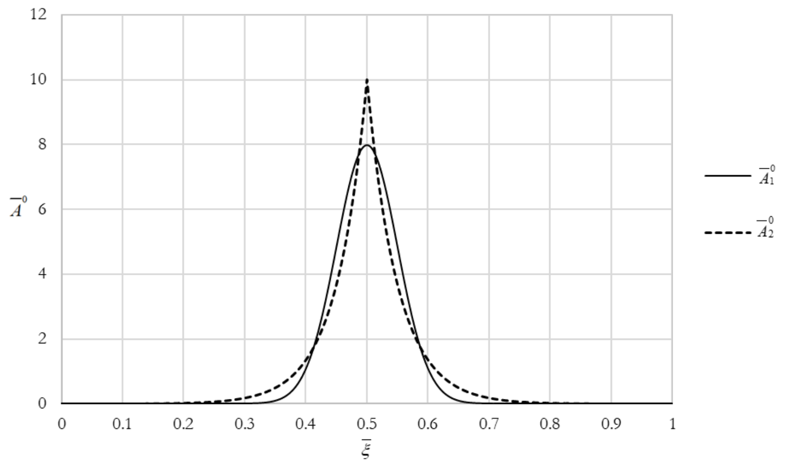

Figure.4 Kernel function versus location along the length of nanocantilever beams ($\overline{\xi}$) at the middle span ($\overline{x}$=0.5) by adopted kernel functions in Eq.(16)-(17) and $\overline{\sigma}$ is given by 0.05

4.2 Buckling load and buckling shape analysis

By comparing both two kernel function shown in figure.5 supposing an equally in size of influenced zone, it can be concluded that the inclusion of nonlocal effect decreases buckling loads which the result is consistent with the available literature of J.N. Reddy [1]. In addition, this study demonstrates how small of cantilever beams that nonlocal effect should be taken into account in analysis of buckling characteristic. However, it depends on acceptance in level of accuracy.

The comparative result shown in figure.5 and 6 demonstrates that both two kernel function seems not to be very sensitive in examining buckling load and fundamental natural frequency of small size dependent nanocantilever beams. Thus, in examining buckling characteristic in the future, researchers may not have to concerned about shape of kernel function, and could choose function which convenient in computation.

According to Figure.7 and 8, representing buckling and vibration shape of cantilever beams which varied by nonlocal parameter $\overline{\sigma}$, the shapes seem to be equal in both behaviors leading to summarize that nonlocal effect does not affect the result in analysis of buckling and free vibration shape. Accordingly, researchers could utilized local theory in investigating buckling and free vibration shape of nanocantilever beams.

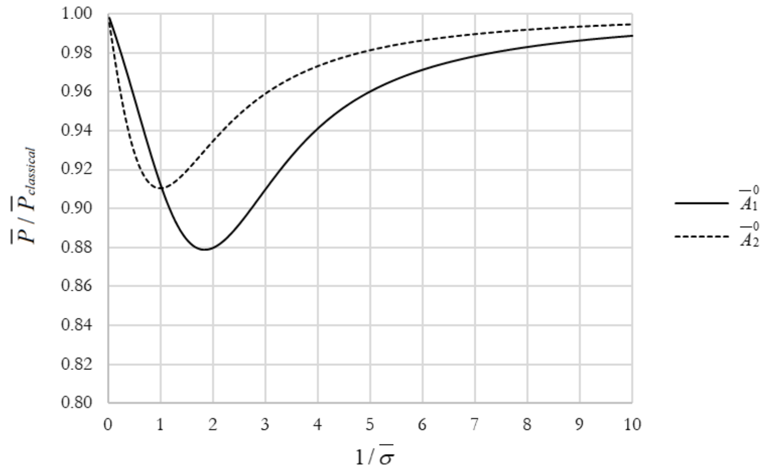

Figure.5 Buckling load determined by nonclassical over classical theory versus inverse of nonlocal parameter in both two kernel functions in Eq.(16)-(17)

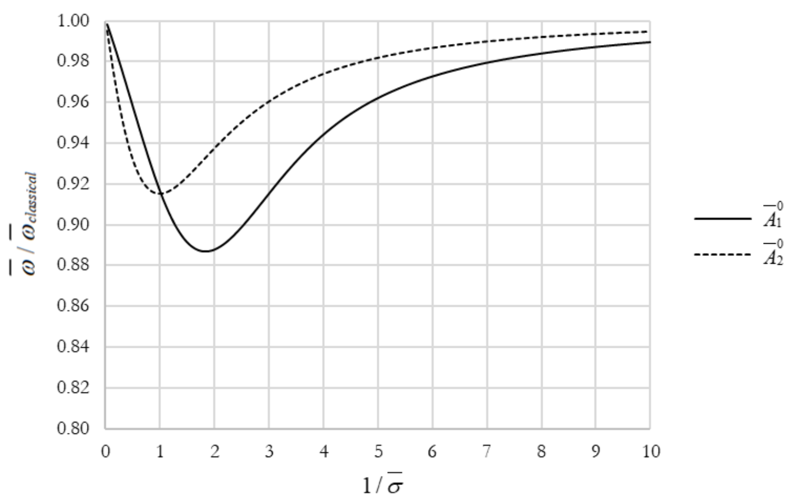

Figure.6 Fundamental natural frequency determined by nonclassical over classical theory versus inverse of nonlocal parameter in both two kernel functions in Eq.(16)-(17)

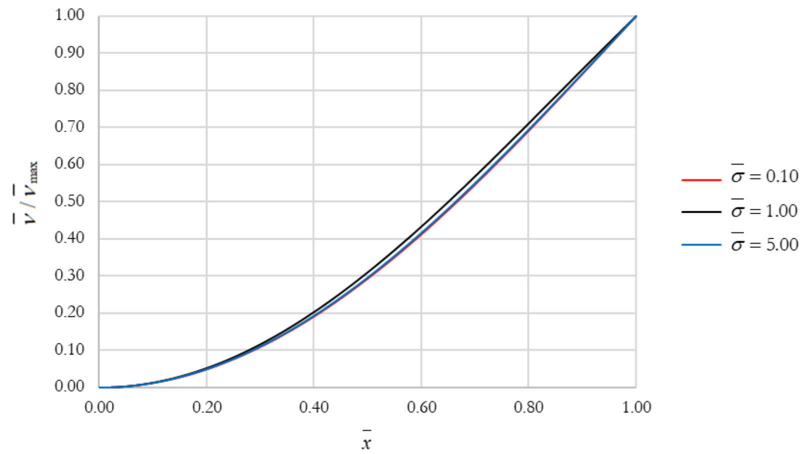

Figure.7 The ratio of deflection over deflection at free end versus location along the length of nanocantilever beams by given $\bar{\sigma}$ equal to 0.1, 1 and 5

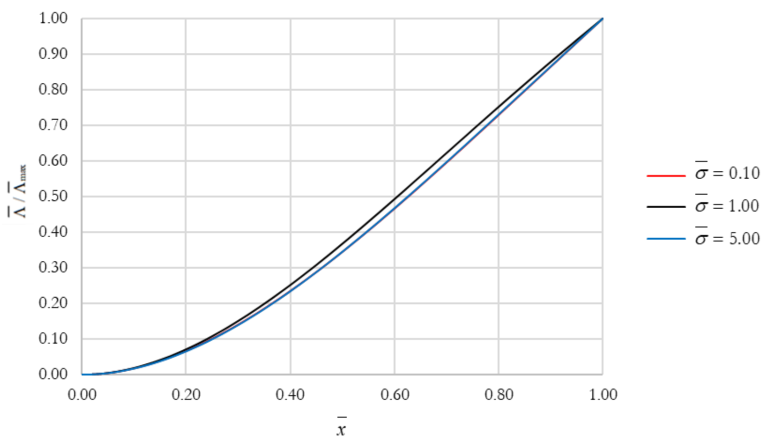

Figure.8 The ratio of amplitude over amplitude at free end versus location along the length of nanocantilever beams by given $\bar{\sigma}$ equal to 0.1, 1 and 5

5. Conclusion and Remarks

In this study, the integral form of Eringen nonlocal equation is presented for both static and dynamic behaviors including buckling and free vibration analysis of Euler-Bernoulli beams which not account for shear and rotary effects and supposing that the area of the nanobeam cross-section is constant throughout element. Galerkin approximation together with Gaussian quadrature techniques are exploited in solving.

The obtained results illustrate that, the inclusion of nonlocal effect decreases both buckling load and fundamental natural frequency compared to local theory. However, nonlocal effect does not very sensitive to buckling shape of cantilever column. Additionally, it has been demonstrated that kernel's shape may not impact the result of buckling load and fundamental natural frequency significantly.

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